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Dirac operator on the q-deformed fuzzy sphere and its spectrum

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Abstract: The q-deformed fuzzy sphere $S^2_{qF}(N)$ is the algebra of $(N+1)\times(N+1)$ dim. matrices, covariant with respect to the adjoint action of $U_q(su(2))$ and in the limit $q \to 1$, it reduces to the fuzzy sphere $S^2_F(N)$. We construct the Dirac operator on the q-deformed fuzzy sphere-$S^2_{qF}(N)$ using the spinor modules of $U_q(su(2))$. We explicitly obtain the zero modes and also calculate the spectrum for this Dirac operator. Using this Dirac operator, we construct the $U_q(su(2))$ invariant action for the spinor fields on $S^2_{qF}(N)$ which are regularised and have only finite modes. We analyse the spectrum for both q being root of unity and real, showing interesting features like its novel degeneracy. We also study various limits of the parameter space (q, N) and recover the known spectrum in both fuzzy and commutative sphere.

Keywords: Quantum Groups, Non-Commutative Geometry, Lattice Quantum Field Theory.

Contents

1. Introduction

In the fuzzy physics programme one of the key objects of interest is the Dirac operator. It is essential for the construction of spinorial actions on fuzzy spaces, giving, thus, kinematics of a fuzzy field. Such field theoretic models on compact fuzzy manifolds are of interest as they provide a regularised theory where the fields have only finite number of modes. Also the fuzzy space being a noncommutative space, Dirac operator is a fundamental object for the study of the noncommutative geometry since it is one of the ingredients required for the construction of the spectral triple [1]. Thus apart from the field theoretic interest, construction and study of the Dirac operator is of intrinsic interest for the construction of differential calculus on the space being considered.

Fuzzy spaces can be seen as highly symmetrical lattices. For this reason we require Dirac operator to satisfy the underlying symmetry of this space so that the field theory models constructed using it will naturally incorporate the symmetry of the underlying space. The prototype example is Dirac operator on the Fuzzy Sphere (S^2_F) [2−6] which is invariant under the SU(2) rotations. In [2], the Dirac operator on fuzzy sphere was constructed and its spectrum and zero modes were obtained. This construction was based on the generalisation of the notions of spinor bundles to their fuzzy analogues. This was then generalised in [3] to include topologically non-trivial spinor modules using the supersymmetric extension of the fuzzy sphere. In this construction, not only interacting fermions but also non-trivial winding number configurations were obtained even at the kinematical level. In [4], using only the bosonic algebra of fuzzy sphere, Dirac operator and its spectrum were obtained. Here the fuzzy sphere algebra was enlarged by including the derivations on fuzzy sphere in order to obtain the chirality operators and Dirac operator having nice commutative limits. Later, exploiting the ambiguity in the operator ordering in this noncommutative generalisation, an alternative chirality operator and the corresponding Dirac operator were obtained in [5] by the same authors and their construction yields a Dirac operator different from that obtained in [3]. In [7] it has been shown that the spinor theory on fuzzy sphere is free of fermion doubling problem and further studies to construct the topological solutions on fuzzy sphere were carried out in [8].

In [9, 10] fuzzy generalisation of CP^2-space was investigated and it was shown that the scalar theory on fuzzy CP^2 is free from UV divergences and the Dirac operator on this space have many interesting features. The investigations to understand the notions of symmetries of noncommutative spaces based on Hopf algebras was attempted in [11]. Studies to construct the noncommutative generalisations of the spaces endowed with symmetries under the action of quantum groups have also been attempted recently [12 − 14]. $U_q(su(2))$ has also been used in the construction of sigma model on fuzzy sphere [15]. A Dirac operator on the $U_q(su(2))$ group manifold was obtained and was shown to have same spectrum as that of the round Dirac operator on a commutative 3−sphere. In [13] Dirac operator and chirality operator on noncommutative space having $U_q(su(2))$ as the symmetry group were constructed. It has been argued that the Dirac operator is *covariant* and in the commutative limit where the underlying space is Podles sphere S^2_q, the full rotational invariance of the Dirac operator is recovered. It was further shown that the Dirac operator reduces to that obtained in [4, 5].

Considering novel symmetries such as those related with quantum groups is a natural follow up in the fuzzy physics programme. The quantum groups have been already appeared in several other physical models such as in Wess-Zumino field theories, string theory and also in knot theory and noncommutative theory [16]. Thus the investigations to see the possibility of constructing field theory models on noncommutative space having invariance under the action of quantum groups are of interest [17]. For the fuzzy physics programme, where one is interested to obtain the regularised field theory with finite number of degrees of freedom, quantum groups, specially the $U_q(su(2))$ is interesting for the introduction of one more parameter, namely q in the theory. This allows one to study different ways of recovering the commutative theory by different limiting procedures.

In this paper we construct a Dirac operator on q-deformed fuzzy sphere-S^2_{qF} described by an algebra $\mathcal{A}(N, q)$. Here q is the deformation parameter and in the limit $q \to 1$, we recover the fuzzy sphere given by the $(N + 1) \times (N + 1)$ matrix algebra $\mathcal{M}(N + 1)$, whose dimension is fixed by the fuzzy parameter N. Here we first construct the $\mathcal{A}(N, q)$ spinor-bimodules by generalising the construction of spinor modules to q-deformed case. The spinor fields are constructed so as to have a natural decomposition of the spinor modules into a direct sum of two submodules which are labelled by plus and minus chirality respectively.

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We also obtain a chirality operator which anti-commutes with the Dirac operator. This is followed by the construction of a pair of operators $K_\pm$ which maps $\pm$ chiral subspace to $\mp$ ones. Using these $K_\pm$, a Dirac operator is constructed. This Dirac operator maps spinor module to itself by construction and also anti-commutes with the chirality operator. We also require the q-deformed, fuzzy Dirac operator and its spectrum to reduce to the known results in various limits. These different limits can be expressed as

$$\begin{aligned}
&S^2_{qF} \to S^2_F \to S^2, \quad (q \to 1 \ \text{ followed by } \ N \to \infty) \\
&S^2_{qF} \to S^2_q \to S^2, \quad (N \to \infty \ \text{ followed by } \ q \to 1) \\
&S^2_{qF} \to S^2, \quad (N \to \infty, \quad q \to 1 \ \text{ simultaneously})
\end{aligned}$$

where S^2_q is the q-deformed sphere and S^2_F is the fuzzy sphere. We show that the spinor modules naturally splits into the direct sum of irreducible representations (IRR) of half-integer spins and this allows one to obtain the eigenvectors and spectrum of Dirac operator. We also obtain the zero modes explicitly. We present a detailed analysis of the spectrum of the Dirac operator showing its interesting, novel behaviour for various ranges of parameters involved (like q and fuzzy cut-off). These novel features are double degeneracy of the spectrum for q being root of unity, level crossing of the spectrum with q and cut-offs on allowed values of topological index which are q dependent. We have also shown that the spectrum in commutative and fuzzy cases are recovered in the appropriate limits.

The present paper is organised as follows. In the next section we present some of the essential results regarding $U_q(su(2))$ as we are using these in deriving our main result. In section 3 we briefly revise the construction of fuzzy spinor module and Dirac operator by Grosse et al [3]. In section 4 we present the main result of the paper, viz; construction of Dirac operator and Chirality operator on q-deformed fuzzy sphere, $\mathcal{A}(N, q)$. We also obtain the spectrum of the Dirac operator and show its zero modes explicitly. In section 5, we analyse the spectrum of deformed Dirac operator on q-deformed fuzzy sphere showing its interesting features and also study its various limits, recovering the fuzzy, deformed and commutative results respectively. Our concluding remarks are given in section 6.

2. $U_q(su(2))$, q-oscillators and Schwinger Realisation

In this section we present the Hopf algebra $U_q(su(2))$ and recall its representation theory [16]. $U_q(su(2))$ is the algebra generated by three operators, $J_\pm, J_3$ satisfying the relations

$$\begin{aligned}
\Big[J_+, J_-\Big] &= \frac{K - K^{-1}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}, \\
\Big[J_3, J_\pm\Big] &= \pm J_\pm
\end{aligned} \tag{2.1}$$

where $K = q^{J_3}$. The Casimir of the algebra is given by

$$C = J_- J_+ + \frac{K q^{\frac{1}{2}} + K^{-1} q^{-\frac{1}{2}}}{(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^2}. \tag{2.2}$$

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The main difference of this algebra with respect to the universal enveloping algebra U($su(2)$) is the co-product. Since we will make use of tensor representation, the co-product, antipode and co-unit play important roles. The co-product, antipode and co-unit for $U_q(su(2))$ are defined to be

$$\Delta(J_\pm) = J_\pm \otimes K^{\frac{1}{2}} + K^{-\frac{1}{2}} \otimes J_\pm \tag{2.3}$$

$$\Delta(J_3) = I \otimes J_3 + J_3 \otimes I \tag{2.4}$$

$$S(J_\pm) = -q^{\pm\frac{1}{2}} J_\pm, S(J_3) = -J_3 \tag{2.5}$$

$$\epsilon(I) = 1, \quad \epsilon(J_\pm) = 0, \quad \epsilon(J_3) = 0 \tag{2.6}$$

The Schwinger realisation of $U_q(su(2))$ algebra in terms of a pair of q-deformed boson operators $A_\alpha, A_\alpha^\dagger, \alpha = 1, 2$ can be expressed as

$$J_+ = A_1^\dagger A_2 \tag{2.7}$$

$$J_- = A_1 A_2^\dagger \tag{2.8}$$

$$J_3 = \frac{1}{2}(N_1 - N_2), \tag{2.9}$$

satisfying $U_q(su(2))$ relations given in eq. (2.1). These q-annihilation and q-creation operators used to define the generators of $U_q(su(2))$ satisfy the following algebra (known as q-Heisenberg algebra)

$$A_\alpha A_\alpha^\dagger - q^{\frac{1}{2}} A_\alpha^\dagger A_\alpha = q^{-\frac{N_\alpha}{2}} \tag{2.10}$$

$$[N_\alpha, A_\alpha^\dagger] = A_\alpha^\dagger \tag{2.11}$$

$$[N_\alpha, A_\alpha] = -A_\alpha. \tag{2.12}$$

Depending on the values of q, the representations of $U_q(su(2))$ can be broadly classified into two main categories. For q taking real values, the representations are simple deformation of those of U($su(2)$) and they are in one-to-one correspondence with each other. For q being root of unity, there are two types of representations. One (which we refer as classical) is similar to that of U($su(2)$), but finite dimensional (whose dimension is fixed by the q). The other is a cyclical representation which is also finite dimensional, but not having any relation to that of U($su(2)$).

In the present work we are interested only in cases where q is root of unity or real, but with classical representation. In this settings we will study certain limits where we recover the usual situation of U($su(2)$).

For both types of representations we are interested, we have,

$$J_\pm|l, m\rangle = \sqrt{[l \pm m + 1]_q [l \mp m]_q}\ |l, m \pm 1\rangle \tag{2.13}$$

$$J_3|l, m\rangle = m|l, m\rangle. \tag{2.14}$$

Here we have used the q-number defined as

$$[n]_q \equiv [n] = \frac{q^{\frac{n}{2}} - q^{-\frac{n}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \tag{2.15}$$

3. Dirac operator on fuzzy sphere

Here we briefly summarise the essential results regarding the construction of spinor modules and $su(2)$ invariant Dirac operator [3] on the fuzzy sphere S_F^2. Apart from fixing our notations, it also serves to compare and contrast the construction of the Dirac operator and its spectrum on fuzzy sphere-S_F^2 and that on q-deformed fuzzy sphere-S_{qF}^2.

The spinor fields on S_F^2 is defined as

$$\Psi = f(a_\alpha^\dagger, a_\alpha)b + g(a_\alpha^\dagger, a_\alpha)b^\dagger. \tag{3.1}$$

Here b and $b^\dagger$ are fermionic annihilation and creation operators and f and g are constructed using a pair of bosonic annihilation and creation operators, $a_\alpha, a_\alpha^\dagger$ alone. Thus both f and g have odd Grassmann parity. The above bosonic and fermionic operators obey $a_1^\dagger a_1 + a_2^\dagger a_2 + b^\dagger b = N$ where N is the fuzzy cut-off parameter.

The the spinor field Ψ given above can be considered as the linear combination of monomials (given below in eq. (3.4)) with fixed topological index $2k = m_1 + m_2 + \mu - n_1 - n_2 - \nu$, $\mu, \nu = 0, 1$ and $k \in \frac{1}{2}\mathbb{Z}$. Here μ, ν represent the number of fermionic creation and annihilation operators appearing in eq. (3.1) and m_1, m_2, n_1, n_2 are non-negative integers. The space of these spinors is denoted as $\mathcal{S}_k, k \in \frac{1}{2}\mathbb{Z}$ and the $su(2)$ algebra has a natural action on these modules. Since $\mu, \nu = 0, 1$ and $\mu + \nu = 1$, we notice that the spinor module naturally splits into two parts

$$\mathcal{S}_k = \mathcal{S}_{k+\frac{1}{2}} b + \mathcal{S}_{k-\frac{1}{2}} b^\dagger \tag{3.2}$$

and f and g belonging to $\mathcal{S}_{k\pm\frac{1}{2}}$ can be generically expressed as

$$\phi = \sum C_{m_1 m_2 n_1 n_2} a_1^{\dagger m_1} a_2^{\dagger m_2} a_1^{n_1} a_2^{n_2}. \tag{3.3}$$

respectively. Here m_1, m_2, n_1, n_2 are non-negative integers.

Note that these bi-modules $\mathcal{S}_k \equiv \mathcal{S}_{M,N}$ ($su(2)$ acts on them by left-right actions) are generated by the monomials of the form

$$a_1^{\dagger m_1} a_2^{\dagger m_2} a_1^{n_1} a_2^{n_2} b^{\dagger \mu} b^\nu \tag{3.4}$$

with $m_1 + m_2 + \mu \leq M, n_1 + n_2 + \nu \leq N, M - N = 2k$ apart from the condition $(m_1 + m_2 + \mu) - (n_1 + n_2 + \nu) = 2k$ and they map the Fock space $\mathcal{F}_N^\nu$ to $\mathcal{F}_M^\mu$. Here, the elements of the space $\mathcal{F}_N^\nu$ are given by

$$|n_1, n_2; \nu >= \frac{1}{\sqrt{n_1! n_2!}} a_1^{\dagger n_1} a_2^{\dagger n_2} b^{\dagger \nu} |0>, \quad n_1 + n_2 + \nu = N \tag{3.5}$$

where $|0>$ is the vacuum state, n_1, n_2 are non-negative integers and $\nu = 0, 1$. Thus we see that any operator $\Phi \in \mathcal{S}_{M,N}$ can be expressed as a $(2N+1) \times (2M+1)$ matrix. From the above discussion, we see that for a given value of k, $f \in \mathcal{S}_{M,N-1}$ and $g \in \mathcal{S}_{M-1,N}$. Also, the spinor operators can be seen to map different Fock space as

$$\begin{aligned} f(a_\alpha^\dagger, a_\alpha)b &: \mathcal{F}_N^1 \to \mathcal{F}_M^0 \\ g(a_\alpha^\dagger, a_\alpha)b^\dagger &: \mathcal{F}_N^0 \to \mathcal{F}_M^1 \end{aligned} \tag{3.6}$$

where the element of $\mathcal{F}_N^\nu$ are defined by eq. (3.5). Thus it is clear that f and g can be expanded in terms of the tensor operators belonging to the half-integer spin representations [3]

$$\frac{M}{2} \otimes \frac{N-1}{2} = |k+\frac{1}{2}| \oplus \ldots\ldots \oplus (J-\frac{1}{2}) \tag{3.7}$$

$$\text{and} \qquad \frac{M-1}{2} \otimes \frac{N}{2} = |k-\frac{1}{2}| \oplus \ldots\ldots \oplus (J-\frac{1}{2}) \tag{3.8}$$

respectively where J is related to M and N by $M+N=2J$.

From the above discussion, we see that the tensor operators belonging to the bi-modules $\mathcal{S}_{k+\frac{1}{2}}$ and $\mathcal{S}_{k-\frac{1}{2}}$, i.e., f and g can be expressed as a linear combination of state vectors corresponding to the IRRs of the half-integer spins ranging from $|k-\frac{1}{2}|$ up to $(J-\frac{1}{2})$ respectively. To this end, we first construct the state vectors for these IRRs of the half-integer spins with arbitrary value of k.

For a given j and k we verify that the lowest weight state is

$$\Phi^j_{J,k,-j} = \mathcal{N} {a_2^\dagger}^{(j+k)} {a_1}^{(j-k)} \tag{3.9}$$

and the state vectors Φ^j_{Jkm} corresponding to other values of m can be obtained by the action of J_+. We also note that fb and $gb^\dagger$ are separately eigenfunctions of the Chirality operator Γ with eigenvalues ± 1. The chirality operator Γ defined using the fermionic operators $b, b^\dagger$ acts on the spinors as

$$\begin{aligned} \Gamma\Psi &= -[b^\dagger b, \Psi] \\ &= f(a_\alpha^\dagger, a_\alpha)b - g(a_\alpha^\dagger, a_\alpha)b^\dagger. \end{aligned} \tag{3.10}$$

This chirality operator allows a natural splitting of the space of spinor fields Ψ (see eq. 3.1) according to chirality. Using the state vectors Φ^j_{Jkm}, we define the $\pm$ chiral spinors as

$$\Phi^{j+}_{J,k,m} = \Phi^j_{J,k+\frac{1}{2},m} b \tag{3.11}$$

$$\Phi^{j-}_{J,k,m} = \Phi^j_{J,k-\frac{1}{2},m} b^\dagger. \tag{3.12}$$

The Dirac operator is defined as an operator mapping the spinor module $\mathcal{S}_k$ to itself and having invariance under U($su(2)$ [3]. It can be written in terms of two operators $K_\pm$ and its action is defined as

$$D\Psi = (K_+ g(a_\alpha^\dagger, a_\alpha)b) + (K_- f(a_\alpha^\dagger, a_\alpha)b^\dagger). \tag{3.13}$$

Here, $K_\pm$ are the operators mapping the spinors from $\pm$ chiral subspace to $\mp$ chiral subspace and their action on a generic vector ϕ are given by

$$K_+\phi = ba_2^\dagger \phi a_1^\dagger b - ba_1^\dagger \phi a_2^\dagger b \tag{3.14}$$

$$K_-\phi = b^\dagger a_1 \phi a_2 b^\dagger - b^\dagger a_2 \phi a_1 b^\dagger \tag{3.15}$$

We also note that the chirality operator Γ anti-commute with the Dirac operator. This guarantees the chiral invariance of the spinor field action constructed using this Dirac operator.

Using the chiral spinors constructed above, we obtain the eigenvectors of Dirac operator $\Psi^{j\pm}_{J,k.m} = \frac{1}{\sqrt{2}}\left[\Phi^{j+}_{J,k,m} \pm \Phi^{j-}_{J,k,m}\right]$ with eigenvalues $\sqrt{(j+\frac{1}{2}+k)(j+\frac{1}{2}-k)}$.

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4. Dirac operator on q-deformed fuzzy sphere

In this section we obtain the Dirac operator on the q-deformed fuzzy sphere. This is done by first constructing the $\mathcal{A}(N,q)$ spinor bi-modules which can be expressed as the direct sum of IRRs of half-integer spins. Here we show the role of the parameter q in deciding the maximal spin along with the fuzzy cut off $\mathcal{N}$. We next show that the eigenfunctions spanning these half-integer spaces, along with fermionic creation and annihilation operators which define a chiral operator, provide a natural splitting of the spinor bi-modules into two submodules characterised by the eigenvalues of chirality operator. We also obtain a pair of operators $K_\pm$ that maps from $\pm$ chiral subspace to $\mp$ chiral subspace and show that the eigenfunctions spanning the half-integer IRRs are also eigenfunctions of $K_\pm$. Using these operators, we then construct the Dirac operator and its eigenfunctions and spectrum are obtained. We also obtain the zero modes of the Dirac operator.

4.1 $U_q(su(2))$ spinor module

The q-deformed spinor belonging to the spinor module $\mathcal{S}_k \equiv \mathcal{S}_{MN}, 2k = (m_1 + m_2 + \mu - n_1 - n_2 - \nu), k \in \frac{1}{2}\mathbb{Z}$ is given as

$$\Psi = f(A^\dagger_\alpha, A_\alpha)b + g(A^\dagger_\alpha A_\alpha)b^\dagger \tag{4.1}$$

where $b, b^\dagger$ are fermionic annihilation and creation operators and $A_\alpha, A^\dagger_\alpha$ are the corresponding q-deformed bosonic operators using which f and g are constructed. The fermionic and q-deformed bosonic operators are related to the fuzzy cut-off parameter $\mathcal{N}$ by the relation $[N_1] + [N_2] + b^\dagger b = \mathcal{N}$, where N_α are the q-number operators. The operators f and g, having odd Grassman parity, can be generically expressed as

$$\phi = \sum C_{m_1 m_2 n_1 n_2} A_1^{\dagger m_1} A_2^{\dagger m_2} A_1^{n_1} A_2^{n_2} \tag{4.2}$$

where m_1, m_2, n_1, n_2 are non-negative integers. Thus the spinor field ψ in eq. (4.1) can be expressed as a linear combination of monomials involving q-creation and q-annihilation operators and *usual* fermionic creation and annihilation operators. These monomials having fixed topological index $2k$ are generically expressed as

$$A_1^{\dagger m_1} A_2^{\dagger m_2} A_1^{n_1} A_2^{n_2} b^{\dagger \mu} b^\nu \tag{4.3}$$

with $m_1 + m_2 + \mu \leq M, n_1 + n_2 + \nu \leq N, M - N = 2k$ and also $(m_1 + m_2 + \mu) - (n_1 + n_2 + \nu) = 2k$ where $k \in \frac{1}{2}\mathbb{Z}$. In the above, μ, ν denote the number of fermionic creation and annihilation operators. Since $\mu, \nu = 0, 1$, and $\mu + \nu = 1$, spinor field naturally decompose into $\pm$ chiral components and the q-tensor operators f and g belong to the bi-modules $\mathcal{S}_{k\pm\frac{1}{2}}$ respectively (see also the discussions after eq. (3.1)).

It is easy to see from the above that the bi-modules $\mathcal{S}_k \equiv \mathcal{S}_{MN}$ satisfy

$$S_{MN} S_{NO} = S_{MO}, \quad S^\dagger_{MN} = S_{NM}$$

as in the former(undeformed) case. These bi-modules S_{MN} maps the Fock spaces $\mathcal{F}^\nu_N \to \mathcal{F}^\mu_M$. These (finite dimensional) Fock spaces are generated by bosonic q-creation operators and fermionic creation operator and is given by

$$|n_1, n_2, \nu >_q = \frac{1}{\sqrt{[n_1]![n_2]!}} A_1^{\dagger n_1} A_2^{\dagger n_2} b^{\dagger \nu} |0>, n_1 + n_2 + \nu = N \tag{4.4}$$

where $|0>$ is the vacuum state annihilated by A_α and b. Thus the operators belonging to S_{MN} can be expressed as $(2N+1) \times (2M+1)$ matrices. The q-tensor operators f and g which forms the spinor in eq. (4.1) belongs to the bi-modules $S_{k \pm \frac{1}{2}}$ and these bi-modules can be written as the direct sum of IRRs corresponding to half-integer spins as

$$\frac{M}{2} \otimes \frac{N-1}{2} = |k + \frac{1}{2}| \oplus \ldots\ldots \oplus (J - \frac{1}{2}) \ \ (\text{for } f) \tag{4.5}$$

$$\text{and} \quad \frac{M-1}{2} \otimes \frac{N}{2} = |k - \frac{1}{2}| \oplus \ldots\ldots \oplus (J - \frac{1}{2}) \ \ (\text{for } g) \tag{4.6}$$

respectively and here $J = \frac{M+N}{2}$ and $k = \frac{M-N}{2}$ as in the undeformed case. Thus the spinor module can be expressed as a sum of IRRs of $U_q(su(2))$. But here q-Clebsh-Gordan coefficients appear in the tensoring of eigenfunctions unlike in the undeformed case where it is governed by the Clebsh-Gordan coefficient. Here we note that by fixing $M + N$ where M and N are the upper cut-off on the total number of creation and annihilation operators, one effectively fixes the fuzzy cut-off. Thus the maximum allowed value of J is restricted by the fuzzy cut-off parameter $\mathcal{N}$.

The set of these spinor fields, $\mathcal{S}_k$ is $U_q(su(2))$-algebra bi-module and thus has a natural action of the algebra on them. This action is defined by the co-products given in eqs. (2.3), (2.4). From this co-product, we verify the action of $U_q(su(2))$ generators $J_\pm$ on an irreducible tensor operator Φ^j_m as

$$\Delta(J_\pm)\Phi^j_m = (J_\pm \Phi^j_m - q^{-\frac{m}{2}} \Phi^j_m J_\pm) q^{-\frac{J_3}{2}} = \sqrt{[j \pm m + 1][j \mp m]}\ \Phi^j_{m\pm 1} \tag{4.7}$$

and using this, we obtain the lowest weight state for each of the above half-integer spin representations. Thus we see, for each given values of j and k, the lowest state is

$$\Phi^j_{J,k,-j} = (A_2^\dagger q^{\frac{N_1}{4}})^{(j+k)} (A_1 q^{-\frac{N_2+1}{4}})^{(j-k)}. \tag{4.8}$$

We can construct the remaining state vectors for all other values of m (for a fixed value if k) by applying the J_+ on this $\Phi^j_{J,k,-j}$ [We can also start from the highest weight state given by $\Phi^j_{J,k,j} = (A_1^\dagger q^{-\frac{N_2}{4}})^{(j+k)} (A_2 q^{\frac{N_1+1}{4}})^{(j-k)}$ and obtain the remaining states by applying J_-].

4.2 Chiral spinors, Chirality operator and $K_\pm$

The components of the fermionic field defined in eq. (4.1), viz; fb and $gb^\dagger$ can be expressed using the above obtained state vectors belonging to IRRs of half-integer spins and the fermionic creation and annihilation operators $b^\dagger$ and b. We can expand these chiral components fb and $gb^\dagger$ in terms of

$$\Phi^{j+}_{J,k,m} = \Phi^j_{J,k+\frac{1}{2},m} b, \ j = |k + \frac{1}{2}|, \ldots, (J - \frac{1}{2}) \tag{4.9}$$

$$\text{and} \quad \Phi^{j-}_{J,k,m} = \Phi^j_{J,k-\frac{1}{2},m} b^\dagger \ j = |k - \frac{1}{2}|, \ldots, (J - \frac{1}{2}) \tag{4.10}$$

respectively. Using the chirality operator Γ having the same form as that in the undeformed case, we see that fb and $gb^\dagger$ defined above are eigenvectors of Γ with eigenvalues ± 1 and its action on spinor is given by

$$\Gamma\Psi = -[b^\dagger b, \Psi]. \tag{4.11}$$

The Dirac operator is required to map the spinor modules $\mathcal{S}_k$ to itself and also required to anti-commute with the above chirality operator. Further, we require the Dirac operator to be invariant in the sense of eq. (4.21). We can construct the Dirac operator satisfying the above requirements if we can provide two operators $K_\pm$ which will swap the elements of $\pm$ chiral subspace to that of $\mp$. That is, the $K_\pm$ operators should be such that

$$K_\pm \Phi^{j\mp}_{J,k,m} \rightarrow \Phi^{j\pm}_{J,k,m} \tag{4.12}$$

and then using these operators, we can construct the Dirac operator with the required properties. These operators, $K_\pm$ satisfying eq. (4.12) are given by

$$K_+\Phi = q^{-\frac{k-m}{4}} q^{-\frac{J_z}{2}} b \left[A_1^\dagger \Phi A_2^\dagger q^{\frac{k}{2}} - A_2^\dagger \Phi A_1^\dagger\right] b \tag{4.13}$$

$$K_-\Phi = q^{-\frac{k+m}{4}} b^\dagger \left[A_1 \Phi A_2 q^{\frac{k}{2}} - A_2 \Phi A_1\right] b^\dagger q^{-\frac{J_z}{2}}. \tag{4.14}$$

Using the above and eq. (4.8), we obtain

$$K_\pm \Phi^j_{J,k,m} = \sqrt{[j \pm k + 1][j \mp k]}\ \Phi^j_{j,k\pm 1,m} \tag{4.15}$$

4.3 Dirac operator and its Spectrum

Next we use these operators $K_\pm$ to construct the Dirac operator which guarantees that the Dirac operator maps spinor module to itself. The action of the Dirac operator is expressed as

$$D\Psi = K_+\Phi^{j-}_{J,k,m} + K_-\Phi^{j+}_{J,k,m}. \tag{4.16}$$

Using the eqs. (4.9), (4.10), (4.15) and (4.16), we obtain,

$$D\Phi^{j\pm}_{J,k.m} = \sqrt{[j + \frac{1}{2} + k][j + \frac{1}{2} - k]}\ \Phi^{j\pm}_{J,k.m} \tag{4.17}$$

Thus we see that the normalised eigenfunctions of the Dirac operator can be written as a linear combination of $\pm$ chiral states as

$$\Psi^{j\pm}_{J,k.m} = \frac{1}{\sqrt{2}}\left[\Phi^{j+}_{J,k,m} \pm \Phi^{j-}_{J,k,m}\right] \tag{4.18}$$

with eigenvalues $\lambda(j,k) = \sqrt{[j + \frac{1}{2} + k][j + \frac{1}{2} - k]}$. From eqs. (4.5) and (4.6), we note that the allowed values of j in the above ranges between $j = |k| + \frac{1}{2}$ to $j = J - \frac{1}{2}$.

We also see that the $|M - N|$ zero modes of Dirac operator are

$$\Psi^{m_1 m_2}_{+0} = A_1^{\dagger m_1} A_2^{\dagger m_2} b^\dagger \tag{4.19}$$

$$\Psi^{n_1 n_2}_{-0} = A_1^{n_1} A_2^{n_2} b. \tag{4.20}$$

For the zero modes ψ_+ the allowed values for the index are $k = \frac{1}{2}(m_1 + m_2 + 1) > 0$ and for Ψ_- it is given by $k = -\frac{1}{2}(n_1 + n_2 + 1) < 0$ and in both cases $j = |k| - \frac{1}{2}$. We also mention that the above spectrum satisfy the index theorem and a detailed analysis of this issue will be reported separatly.

4.4 Spinor and $U_q(su(2))$ Invariance

Using the co-product defined in eqs. (2.3), (2.4), (4.7) and eq. (4.15), it is easy to see that the chiral spinors defined in eqs. (4.9) and (4.10) obey

$$\Delta J_{\pm}(K_{\pm}\Phi) - K_{\pm}(\Delta(J_{\pm})\Phi) = 0. \tag{4.21}$$

To give an explicit example, we consider the simplest case of $j = \frac{1}{2} = k$ where the states are $\Phi^{\frac{1}{2}}_{J,\frac{1}{2},-\frac{1}{2}} = A_2^{\dagger} q^{\frac{N_1}{4}}, \Phi^{\frac{1}{2}}_{J,\frac{1}{2},\frac{1}{2}} = A_1^{\dagger} q^{\frac{-N_2}{4}}$ and $j = \frac{1}{2} = -k$ for which the states are $\Phi^{\frac{1}{2}}_{J,-\frac{1}{2},-\frac{1}{2}} = -A_1 q^{\frac{-(N_2+1)}{4}}, \Phi^{\frac{1}{2}}_{J,\frac{1}{2},\frac{1}{2}} = A_2 q^{\frac{(N_1+1)}{4}}$. Here the $J_{\pm}$ operators act (through the co-product) as raising and lowering operators for $k = \frac{1}{2}$ and $k = -\frac{1}{2}$ sets separately and $K_{\pm}$ maps from one set to another but keeping the m value unaltered. This property of Φs given in eq. (4.21) guarantees the invariance of the action constructed using the spinor fields Ψ as we will see below.

The spinor field Ψ can be expressed as a linear combination of the zero modes and $\pm$ chiral spinors as

$$\Psi = \sum_{m_1,m_2} C^{\pm}_{m_1 m_2} \Psi^{m_1 m_2}_{\pm 0} + \sum_{j=|k|+\frac{1}{2}}^{J-\frac{1}{2}} \left(C^{j+}_{km} \Phi^{j+}_{J,k,m} + C^{J-}_{km} \Phi^{j-}_{J,k,m} \right) \tag{4.22}$$

where the Cs are coefficients. Using this Ψ and $\Psi^{\dagger}(= bg^{\dagger} - b^{\dagger} f^{\dagger})$ and the Dirac operator, the spinor action can be defined as shown below.

4.5 Invariant Trace and Spinor Action on S^2_{qF}

The invariant action for the spinorial field on fuzzy sphere is defined using the trace-Tr which is invariant under the action of underlying algebra-$su(2)$. In the case of the fuzzy sphere this trace was the usual trace of matrices. In the present case where the symmetry algebra is $U_q(su(2))$, the trace defined in this way is not invariant. Thus we need to define a Trace operation satisfying

$$Tr(\Delta(a)\hat{A}) = \epsilon(a) Tr(\hat{A}), \tag{4.23}$$

where a are the generators of $U_q(su(2))$ and ϵ is the co-unity defined in eq. (2.6). Here $\hat{A}$ is a matrix defined using the tensor operators of $U_q(su(2))$ as

$$\hat{A} = \sum_{j=0}^{N} \sum_{k=-j}^{j} A_{jk} \hat{T}_{jk}. \tag{4.24}$$

The non-invariance of the usual trace is because of the co-product appearing in eq. (4.23). Following the general prescription of the deformation of the trace [19] we define a new trace which is invariant under $U_q(su(2))$. For this, instead of $\hat{A}$ belonging to the representaion space $\mathcal{H}_l \otimes \mathcal{H}_l^*$, we consider the matrix $\hat{A}^b$ in the equivalent representation space $\mathcal{H}_l^{**} \otimes \mathcal{H}_l^*$. Thus define prescription for the $q-$trace as

$$Tr_q(\hat{A}) = Tr(\hat{K}\hat{A}) \tag{4.25}$$

where $\hat{K}$ is the matrix representation of $K = q^{J_3}$. It can be easily verified that the above trace satisfies the invariance condition given in eq. (4.23). Using this invariant trace, the action for spinorial fields with topological index k is given by:

$$S_{2k} = \frac{2\pi R^2}{[N+1]} Tr_q \left[\bar{\Psi} D \Psi + V(\bar{\Psi}\Psi) \right], \tag{4.26}$$

where $\frac{2\pi R^2}{[N+1]}$ is a normalization factor, with R being associated with the radius of the underlying sphere. For arbitrary topological index we have:

$$S = \sum_{2k} S_{2k} \tag{4.27}$$

Thus, using the new definition of the trace, we obtain an $U_q(su(2))$ invariant action (4.26) for spinorial fields.

5. Spectrum and its Limits

In this section, we analyse the spectrum of the Dirac operator on the q-deformed fuzzy sphere and study its various limits.

The spectrum of the Dirac operator given in eq. (4.17) depends on j and k. But unlike the commutative and/or fuzzy cases, here it depends on q also. Here we analyse the response of the spectrum to changes in these parameters. We also study various limits of this spectrum. We consider two main classes, viz; q being root of unity and real. In both these cases, we also study the behaviour of the spectrum for both vanishing and non-vanishing topological index. Since the eigenvalues for both chiral spinors $\Phi^{j\pm}_{J,k.m}$ are same, without lose of generality, we consider only the positive values of $2k$.

5.1 The case of q root of unity

First we consider the case where q is root of unity, i.e., $q = e^{\frac{2\pi i}{p}}, p \in \mathbb{Z}$. Using this in eq. (2.15), we see that

$$[x] = \frac{sin(\frac{\pi x}{p})}{sin(\frac{\pi}{p})}. \tag{5.1}$$

From this, it is clear that $[N+1] = 0$ if $p = N+1$. This shows that the fuzzy cut-off parameter N (which fixes the matrix dimension of the operators to be $(N+1) \times (N+1)$) has to satisfy the constraint

$$N + 1 < p. \tag{5.2}$$

With this condition on q, we first consider the spectrum for topological index, $2k = 0$ case. In figure (1) we show the variation of the spectrum for varying j (for different values of p). From the figure (1) we notice that spectrum is not monotonically increasing, unlike in the case of the Fuzzy SphereS^2_F (where $Spec(D_F, k = 0) = j$). The striking feature of the spectrum is its *new* degeneracy (apart from the one with respect to the quantum number m) which is not shared either by fuzzy or by commutative counterparts. This can be easily seen by considering the plot of spectrum in figure (1) for $p = 9$, for example. In this case

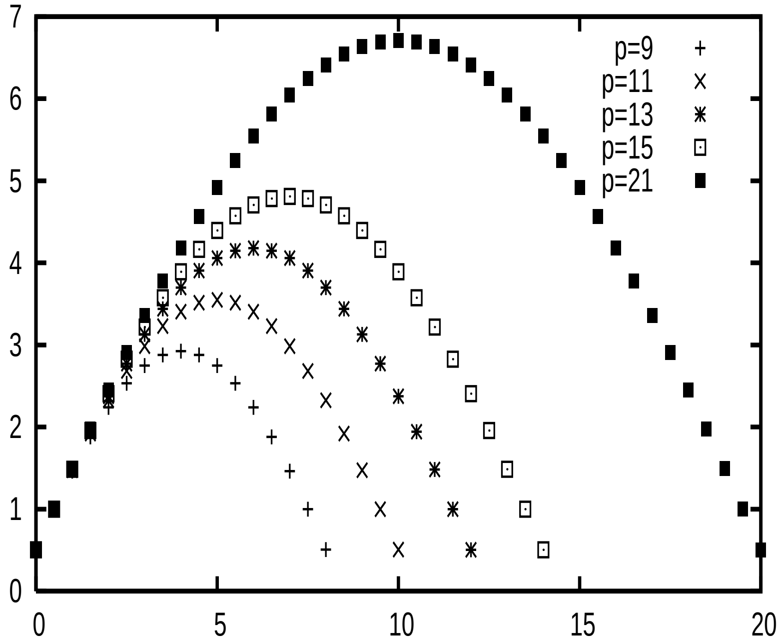

Figure 1: Spectrum for $q = e^{\frac{2\pi i}{p}}$ and $2k = 0$ as a function of spin j for different values of p.

Degenerate levels for $p = 9$	Value of Spectrum
$\lambda(0,0) = \lambda(8,0)$	0.507713
$\lambda(\frac{1}{2},0) = \lambda(\frac{15}{2},0)$	1.000000
$\lambda(1,0) = \lambda(7,0)$	1.41902
$\lambda(\frac{3}{2},0) = \lambda(\frac{13}{2},0)$	1.879385
$\lambda(2,0) = \lambda(6,0)$	2.239764
$\lambda(\frac{5}{2},0) = \lambda(\frac{11}{2},0)$	2.532089
$\lambda(3,0) = \lambda(5,0)$	2.747477
$\lambda(\frac{7}{2},0) = \lambda(\frac{9}{2},0)$	2.879385
$\lambda(4,0)$	2.923804

Table 1: Eigenvalues $\lambda(j,k)$ for different values of j with $k = 0$

the maximum value N can take is 7 and we see for j taking values up to $\frac{(N+1)}{2}$, there is no 'degeneracy' in the spectrum. But as soon as j is allowed to take values above $\frac{(N+1)}{2}$, *new* degeneracy sets in. Thus, in general we see that for a fixed p (and $N < p-1$), the spectrum becomes *doubly* degenerate once $N > \frac{p-1}{2}$. For the example of $p = 9$, we show this double degneracy explicitly in the table-I where the eigenvalues of Dirac operator $\lambda(j,k)$ is given for different values of j with $k = 0$.

In the next graph (figure 2), we plot the spectrum for given values of j but as a function

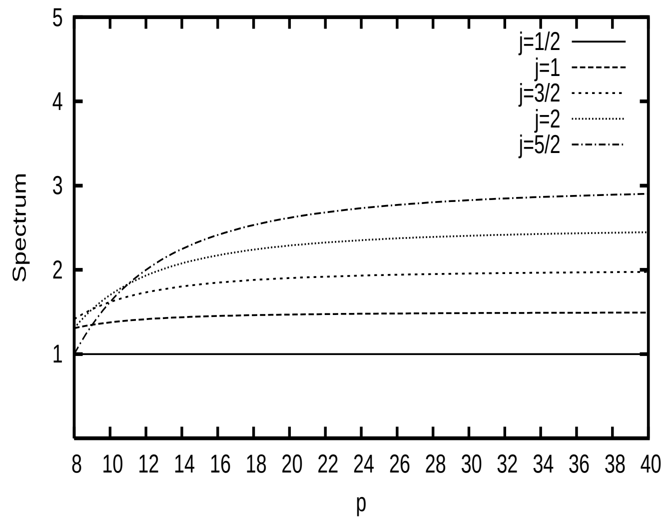

Figure 2: Variation of the spectrum with p for different values of j for $q = e^{\frac{2\pi i}{p}}$ and $2k = 0$.

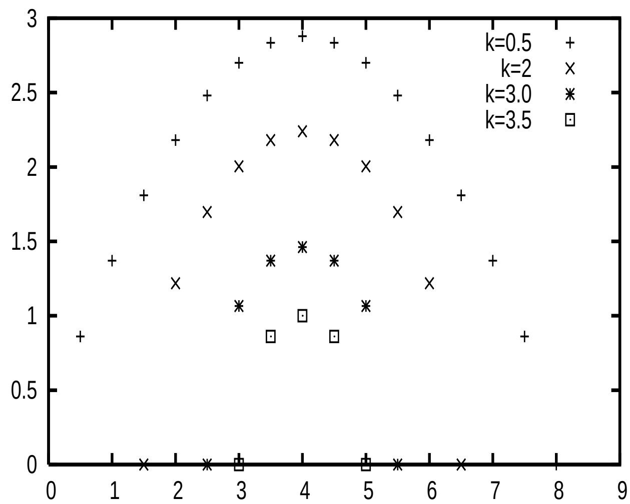

Figure 3: Variation of the spectrum with j for different values of k for $q = e^{\frac{2\pi i}{p}}$ and $p = 16$.

of p. For small values of p, we see the spectrum has a different behaviour and it shows a growth with increasing p. For small j but large p, the spectrum behaves very similar to that of the fuzzy case. From this graph, we also note that the spectrum corresponding to the higher values of j lie below that of smaller j when p is low. As p increases, there is a *level crossing* and the spectrum for higher j raise above that of smaller j.

Next we plot the variation spectrum with j, but for fixed, non-vanishing topological indices $2k$ in (figure 3). We notice that the spectrum is degenerate for non-vanishing values of k also. More interestingly we see that the spectrum becomes zero for certain values of j for each given values of k. But since we are dealing with non-zero modes only, this imply

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that all values of k are not available for every spin j. The restrictions on k will become clear below where we analyse various limits of the spectrum.

First we consider the limit of $p \to \infty$ with N fixed. Since in the limit $p \to \infty$ we have $q \to 1$ and the q-number goes to the usual number,i.e., $[x] \to x$. Thus we expect to recover the spectrum of fuzzy Dirac operator and we see below that this is true. Considering first the case for vanishing topological index, the eigenvalue equation for the q-Dirac operator

$$D^q \Phi^{q,j}_{J,0,m} = [j + \frac{1}{2}] \Phi^{q,j}_{J,0,m}, \tag{5.3}$$

showing that for the highest value of j, the eigenvalue is $[\frac{N}{2} + \frac{1}{2}]$. Now, taking the limit $p \to \infty$ with N fixed, it becomes $(\frac{N}{2} + \frac{1}{2})$, giving the result of fuzzy case. It is easy to see that the same happens for every value of $j < N$.

Next, for the case of non-vanishing topological index, consider the eigenvalue equation given in eq. (4.17). Since we are not considering the zero-modes, each of the factors in the eigenvalue have to be non-zero which lead to the inequalities

$$\frac{N+1}{2} - k \neq 0 \tag{5.4}$$

$$\frac{(N+1)}{2} + k < p, \tag{5.5}$$

eq. (5.4) sets a lower cut-off for the allowed values of N for fixed k. For a fixed $N(=\beta)$, eq. (5.5) and eq. (5.2) together imply $k < p - \frac{(\beta+1)}{2}$. Since the highest allowed value of β satisfying eq. (5.2) is $\beta = p - 2$, we get

$$k < \frac{(p+1)}{2}. \tag{5.6}$$

Thus we see here that p acts as an upper cut-off for the topological index $2k$. Thus by fixing p, we not only impose an upper cut-off for the fuzzy parameter N but also restrict the allowed values for topological index. This feature is particular to q-deformed fuzzy sphere with q being root of unity.

We also note that, as in the $k = 0$ case, here too, in the limit of $p \to \infty$ with fixed N, the spectrum becomes that of the fuzzy Dirac operator.

Next we consider the case where $N \to \infty$, $p \to \infty$, $\frac{N}{p} = \alpha$ (say, with $0 < \alpha < \frac{1}{2}$). In this way, one passes from the q-deformed fuzzy sphere to the commutative sphere. Since $p \to \infty$ imply $q \to 1$ and thus in this limit, the spectrum goes to that of the commutative sphere as the fuzzy cut off N is sent to ∞. This happens for both zero and non-zero values of $2k$.

5.2 The case of the q being real

Here we study the case when the deformation parameter q is real, i.e., $q = e^{2p}$. In this case, from eq. (2.15) we see that the q-number can be expressed as

$$[x] = \frac{sinh(xp)}{sinh(p)}. \tag{5.7}$$

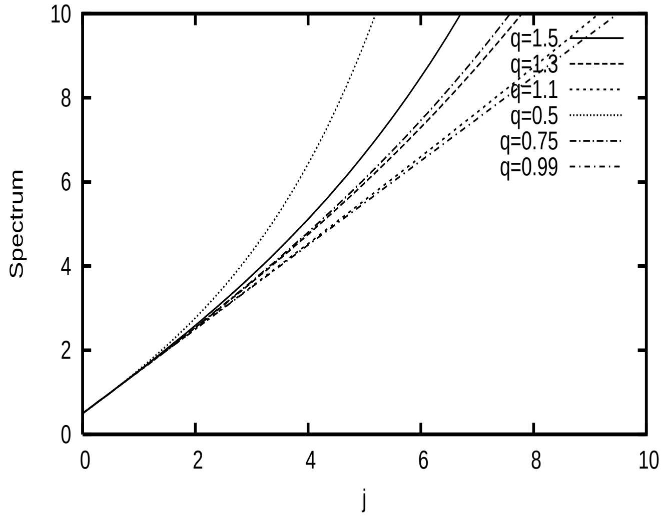

Figure 4: Spectrum as a function of j for different values of real q and $2k = 0$.

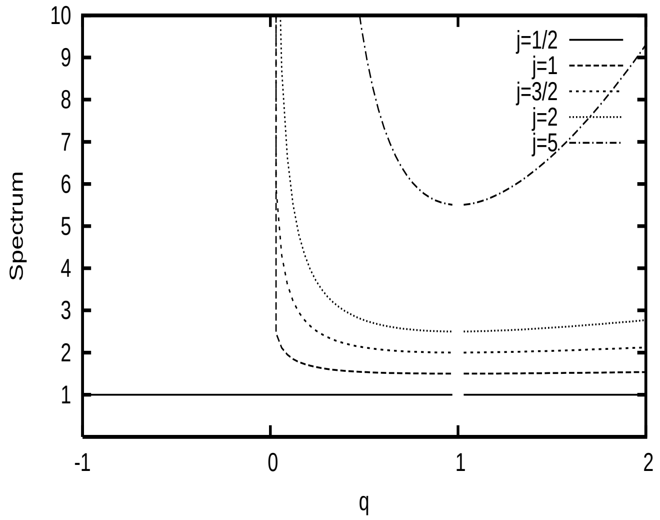

Figure 5: Spectrum as a function of (real) q for different values of j and $2k = 0$.

Thus, we see that p does not introduce any restriction on the fuzzy cut-off N, unlike in the situation where q was root of unity. Thus in the present case, the cut-off N can be arbitrarily large and thus we can take the limit of $N \to \infty$ independently of the parameter q.

In figure (4) we show the variation of the spectrum with change in the spin j (for fixed values of q). We see that the spectrum is not doubly degenerate unlike in the previous case(where $q = e^{\frac{2\pi i}{p}}$). From these two plots, we see clearly that as $q \to 1$, the spectrum approaches the fuzzy sphere result.

In figure (5), we show the variation of the spectrum with q for different values of j.

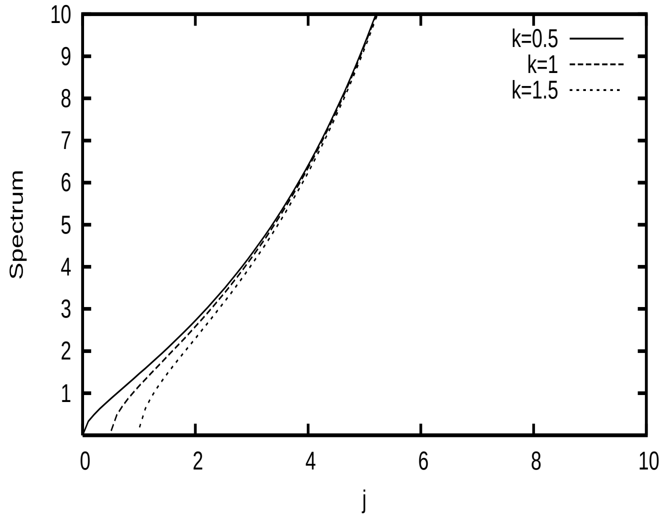

Figure 6: Spectrum as a function of j for $k \neq 0$ for real $q, q = 2$.

We note that the level crossing observed in figure (2) for q root of unity is absent for this case of real q.

In figure (6), we plot the spectrum against j, for given values of k. We see that the spectrum becomes zero for certain values of j. Since we are considering the non-zero modes, this imply restrictions on allowed values of k as in the previous case of $q = e^{\frac{2\pi i}{p}}$.

Here too, the limits we are interested are $N \rightarrow \infty$, $q \rightarrow 1$ giving first q-deformed and then commutative spheres respectively. As in the previous case, by sending the fuzzy cut-off to infinity, we allow all values of spins and thus we recover the spectrum on q-deformed sphere. Now setting the limit $q \rightarrow 1$, we get the spectrum to be $\sqrt{(j+\frac{1}{2}+k)(j-\frac{1}{2}+k)}$ which is that of commutative sphere. Whereas by keeping N fixed and taking $q \rightarrow 1$ results the spectrum corresponding to the fuzzy sphere Dirac operator.

Thus we see from the above plots that for both $q = e^{\frac{2\pi i}{p}}$ and $q = e^{2p}$, we recover the known spectrum on fuzzy as well as commutative limits for both zero and non-zero values of topological index.

6. Conclusions

In this paper, we have constructed $U_q(su(2))$ invariant Dirac operator on q-deformed fuzzy sphere and obtained its spectrum. This construction is based on the realisation of spinor bi-modules using the creation and annihilation operators of fermionic oscillator and a pair of q-deformed bosonic operators. We have also shown that the chirality operator constructed using the fermionic operators naturally splits the spinor space into $\pm$ chiral subspaces. After expressing these subspaces as the direct sum of IRRs of half-integer spins, we have derived the operators $K_{\pm}$ which maps the states of one chiral subspace to other. Using this $K_{\pm}$, we have constructed the Dirac operator which maps the spinor space to itself and anti-commute with the chirality operator as required. We obtain the eigenspinors using the

$U_q(su(2))$ eigenstates belonging to the IRRs of half-integer spins. Using the well defined action of $K_\pm$ on these states, we have calculated the spectrum of the Dirac operator. We also obtained the zero modes explicitly.

The Dirac operator is constructed here using the operators $K_\pm$ which maps the $\pm$ chiral subspaces to $\mp$ chiral subspaces. The form of these operators are non-trivial compared to that in the undeformed (i.e., $q = 1$) case. Here we note that these operators in the case of commutative sphere (see [3]) are constructed using the differential operators with respect to the co-ordinates of the underlying (commutative) sphere. Thus, it is natural to expect these derivatives to be replaced with the q-derivatives when going to the q-sphere (which is related to S^2_{qF} we considered above). It is well known that the q-derivatives have highly non-trivial and asymmetric form compared to one in the commutative space [16, 18]. This fact is reflected in our operators $K_\pm$ also. We stress the fact that the $K_\pm$ reduces to correct fuzzy operators in the limit $q \to 1$. We also notice, using eq. (4.15), that the q-deformed, fuzzy Laplacian [20] expressed in terms of $K_\pm$ gives

$$\begin{aligned}\frac{1}{2}(K_+K_- + K_-K_+)\Phi^j_{jkm} &= \frac{1}{2}([j+1+k][j-k] + [j+1-k][j+k])\Phi^j_{Jkm} \\ &= ([j][j+1] + [k]^2 q^{\frac{2j+1}{2}} + (q^{\frac{1}{2}} - q^{-\frac{1}{2}})[k]^2([j]^2 q^{\frac{j+1}{2}} + [j+1]q^{\frac{j}{2}})\Phi^j_{Jkm} \quad . \qquad (6.1)\end{aligned}$$

The last two terms on the right are known as Lichnorowicz correction to the Laplacian. The above expression shows that in the limit $q \to 1$, the $\frac{1}{2}(K_+K_- + K_-K_+)$ correctly reproduces the Laplacian operator. Thus we also see that the $K_\pm$ leads to a q-deformed, fuzzy Laplacian having correct commutative limits.

We have analysed the spectrum of the Dirac operator and its various limits. We showed that the known results are recovered in the appropriate limits. We also showed that the spectrum of the q-deformed fuzzy Dirac operator has many novel interesting features. We have shown that the spectrum can be doubly degenerate depending on the value of p for the case of q being root of unity. We have also obtained restrictions on the fuzzy parameter and topological index coming from the deformation parameter q for the case of q being root of unity. In both q real as well as root of unity, we found that the allowed values of the topological index are constrained by the deformation parameter q as well as by fuzzy cut-off N.

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Note added in proof: a different construction of Dirac operator on q-deformed sphere was carried out in [21] and a general scheme to construct Dirac operator for coset spaces have been developed in [22]. We thank H. Steinacker for discussions and brining the first reference to our notice.

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